

# Neutrino oscillations in the scheme of mass mixings and problem of smallness of angle mixing $\theta_{13}$

Kh. M. Beshtoev

Joint Institute for Nuclear Research, Joliot Curie 6, 141980 Dubna,  
Moscow region, Russia.

## Abstract

In the framework of the mass mixing scheme we have considered mixings and oscillations of  $\nu_e, \nu_\mu, \nu_\tau$  neutrinos and obtained expressions for angle mixings and lengths of oscillations in dependence on components of the nondiagonal mass matrix. Then analysis of these obtained results was done by using modern experimental data on neutrino oscillations. It has been shown that in this approach the lengths of neutrino oscillations  $L_{23}$  and  $L_{13}$  are not compulsory to be equal. It means that the angle mixing  $\theta_{13}$  can be not very small, i.e.,  $L_{13}$  can be larger than  $L_{23}$ .

In the conventional approach  $L_{13} \approx L_{23}$  ( $L_{12} \gg L_{23}$ ) and angle mixing of  $\theta_{13}$  is very small. Angle mixings  $\theta_{23}, \theta_{12}$  are big. Then there is a problem: why is mixing angle  $\theta_{13}$  so small? A natural solution of the problem is to suppose that  $(m_2^2 - m_1^2) \neq (m_3^2 - m_1^2) - (m_3^2 - m_2^2)$ , then  $L_{13} > L_{23}$ . It will be realized if there are 4 neutrino oscillations instead of 3 neutrino oscillations. Then the value of  $\theta_{13}$  is necessary to search at distances more than  $L_{23}$ .

## 1 Introduction

Oscillations of  $K^0$  mesons (i.e.,  $K^0 \leftrightarrow \bar{K}^0$ ) were theoretically [1] and experimentally [2] investigated in the 50-s and 60-s. Recently an understanding has been achieved that these processes go as a double-stage process [3, 4, 5, 6]. A detailed study of meson mixing and oscillations is very important since the theory of neutrino oscillations is built in analogy with the theory of meson oscillations.

The suggestion that, in analogy with  $K^o, \bar{K}^o$  oscillations, there could be neutrino-antineutrino oscillations ( $\nu \rightarrow \bar{\nu}$ ), was considered by Pontecorvo [7] in 1957. It was subsequently considered by Maki et al. [8] and Pontecorvo [9] that there could be mixings (and oscillations) of neutrinos of different flavors (i.e.,  $\nu_e \rightarrow \nu_\mu$  transitions).

Lengths of three neutrino oscillations are [10] as follows

$$L_{12} = 2\pi \frac{2p}{|m_2^2 - m_1^2|}, \quad L_{13} = 2\pi \frac{2p}{|m_3^2 - m_1^2|}, \quad L_{23} = 2\pi \frac{2p}{|m_3^2 - m_2^2|}. \quad (1)$$

In experiments [11, 12]  $L_{12}$  and  $L_{23}$  were measured and it was obtained that  $L_{12} \gg L_{23}$ . If to use the expression

$$(m_2^2 - m_1^2) = (m_3^2 - m_1^2) - (m_3^2 - m_2^2), \quad (2)$$

taking into account that  $m_2^2 - m_1^2 = \frac{4\pi p}{L_{12}}$ ,  $m_3^2 - m_1^2 = \frac{4\pi p}{L_{13}}$ ,  $m_3^2 - m_2^2 = \frac{4\pi p}{L_{23}}$ , then we obtain

$$L_{13} = \frac{L_{12}L_{23}}{L_{12} + L_{23}}. \quad (3)$$

Since  $L_{12} \gg L_{23}$ , then from expression (3) we get

$$L_{13} \approx L_{23}. \quad (4)$$

The mixing angle  $\theta_{13}$  measured in experiment [11] is very small. Then there is a question: why are mixing angles  $\theta_{23}$  and  $\theta_{12}$  measured in the experiment of the order of maximal angle ( $\pi/4$ ) and why is the measured angle  $\theta_{13}$  is so small? There is definitely a problem. To investigate this problem, we will work in the framework of the masses mixing scheme.

Since the scheme (theory) of neutrino oscillations is constructed in analogy with  $K^o$  mesons, at first we consider the scheme of  $K^o$  meson oscillations and then the scheme of neutrino oscillations.

## 2 Vacuum mixings and oscillations of $K^o, \bar{K}^o$ mesons at strangeness violation by weak interactions

$K^o$ -,  $\bar{K}^o$ - meson states are produced in the strong interaction (i.e, they are eigenstates of these interactions), then the mass matrix of  $K^o$  -

mesons will have a diagonal form [3, 4, 5, 6]. Following the traditions we will consider the  $K^o$  - meson mixings and oscillations by using the mass matrix and for convenience the masses are used in the linear but not in the quadratic form, then the mass matrix has the following form:

$$\begin{pmatrix} m_{K^o} & 0 \\ 0 & m_{\bar{K}^o} \end{pmatrix}. \quad (5)$$

Because of the weak interactions violating strangeness ( $s \leftrightarrow d$ ) this mass matrix (5) becomes a nondiagonal matrix:

$$\begin{pmatrix} m_{K^o} & m_{K^o\bar{K}^o} \\ m_{\bar{K}^o K^o} & m_{\bar{K}^o} \end{pmatrix}. \quad (6)$$

For obtaining the eigenstates of weak interactions which violate strangeness, we have to diagonalize this matrix by turning on angle  $\theta_o$ :

$$\begin{pmatrix} m_{K^o} & m_{K^o\bar{K}^o} \\ m_{\bar{K}^o K^o} & m_{\bar{K}^o} \end{pmatrix} \rightarrow U^{-1} \begin{pmatrix} m_{K_1^o} & 0 \\ 0 & m_{K_2^o} \end{pmatrix} U, U = \begin{pmatrix} \cos\theta_o & -\sin\theta_o \\ \sin\theta_o & \cos\theta_o \end{pmatrix}. \quad (7)$$

By using this procedure, we get

$$\begin{aligned} \tan 2\theta_o &= \frac{2m_{K^o\bar{K}^o}}{|m_{K^o} - m_{\bar{K}^o}|}, \\ \sin 2\theta_o &= \frac{2m_{K^o\bar{K}^o}}{\sqrt{(m_{K^o} - m_{\bar{K}^o})^2 + (2m_{K^o\bar{K}^o})^2}}, \\ m_{1,2} = m_{K_1, K_2} &= \frac{1}{2} \left[ (m_{K^o} + m_{\bar{K}^o}) \pm ((m_{K^o} - m_{\bar{K}^o})^2 + 4m_{K^o\bar{K}^o}^2)^{1/2} \right], \end{aligned} \quad (8)$$

where  $K_1^o$  and  $K_2^o$  states are eigenstates of the weak interactions violating strangeness. Now these states are superposition states of  $K^o, \bar{K}^o$  mesons

$$\begin{aligned} K_1^o &= \cos\theta_o K^o - \sin\theta_o \bar{K}^o, \\ K_2^o &= \sin\theta_o K^o + \cos\theta_o \bar{K}^o, \end{aligned} \quad (10)$$

and the inverse transformation gives the following:

$$\begin{aligned} K^o &= \cos\theta_o K_1^o + \sin\theta_o K_2^o, \\ \bar{K}^o &= -\sin\theta_o K_1^o + \cos\theta_o K_2^o. \end{aligned} \quad (11)$$

The computation of nondiagonal components of the mass matrix (6) can be fulfilled by using the Feynman block diagrams in the framework of the standard model of electroweak interactions [13], [14] with Kabibbo-Kobayashi-Maskawa [15] matrices.

Expression for  $\sin^2 2\theta_o$  can be obtained from (8) and it is given by the following expression ( $\theta_o$  is the angle of mixing):

$$\sin^2 2\theta_o = \frac{(2m_{K^o\bar{K}^o})^2}{(m_{K^o} - m_{\bar{K}^o})^2 + (2m_{K^o\bar{K}^o})^2}. \quad (12)$$

This expression has the analogy with a well knowing formula of Breit-Wigner [16] for transition probability,

$$W(m, m_o, \Gamma, \dots) = \frac{c(\frac{\Gamma}{2})^2}{(m - m_o)^2 + (\frac{\Gamma}{2})^2}, \quad (13)$$

where  $c$  - is a normalized factor,  $m, m_o$  are masses and  $\Gamma$  is the width transition. We can obtain expression (12) from Breit-Wigner formula if we fulfill the following substitutions:

$$m \rightarrow m_{K^o}, \quad m_o \rightarrow m_{\bar{K}^o}, \quad \frac{\Gamma}{2} \rightarrow 2m_{K^o\bar{K}^o} \quad c \rightarrow 1. \quad (14)$$

We see that we can interpret nondiagonal mass term  $2m_{K^o\bar{K}^o}$  as half-width of  $K^o \leftrightarrow \bar{K}^o$  transitions.

Since the weak interactions are  $CPT$  invariant, then  $m_{K^o K^o} = m_{\bar{K}^o \bar{K}^o}$  and the mixing angle  $\theta_o$  will be equal to  $\frac{\pi}{4}$ , i.e.,  $\sin^2 2\theta_o = 1$ . Then from expression (14) and (11) we get

$$K_1^o = \frac{K^o - \bar{K}^o}{\sqrt{2}}, \quad K_2^o = \frac{K^o + \bar{K}^o}{\sqrt{2}}, \quad (15)$$

$$K^o = \frac{K_1^o + K_2^o}{\sqrt{2}}, \quad \bar{K}^o = \frac{K_1^o - K_2^o}{\sqrt{2}}. \quad (15')$$

It is necessary to remark that  $CPK_1^o = K_1^o$  and  $CPK_2^o = -K_2^o$ , i.e.,  $CP$  parity  $K_1^o$  meson is a positive value and it can decay into two  $\pi$  mesons, and  $CP$  parity of  $K_2^o$  meson is a negative value and it can decay into three  $\pi$  mesons.

The evolution of  $K_1^o, K_2^o$  meson states with masses  $m_1, m_2$  will be given with the following expression:

$$K_1^o(t) = e^{-iE_1 t} K_1^o(0), \quad K_2^o(t) = e^{-iE_2 t} K_2^o(0), \quad (16)$$

where

$$E_k^2 = (p^2 + m_k^2), k = 1, 2.$$

If these mesons are moving without interactions, then

$$\begin{aligned} K^o(t) &= \cos\theta_o e^{-iE_1 t} K_1^o(0) + \sin\theta_o e^{-iE_2 t} K_2^o(0), \\ \bar{K}^o(t) &= -\sin\theta_o e^{-iE_1 t} K_1^o(0) + \cos\theta_o e^{-iE_2 t} K_2^o(0). \end{aligned} \quad (17)$$

Using expression (10) for  $K_1^o, K_2^o$  and putting them in (17), we obtain

$$\begin{aligned} K^o(t) &= [e^{-iE_1 t} \cos^2\theta_o + e^{-iE_2 t} \sin^2\theta_o] K^o(0) + \\ &\quad + [e^{-iE_1 t} - e^{-iE_2 t}] \sin\theta_o \cos\theta_o \bar{K}^o(0), \\ \bar{K}^o(t) &= [e^{-iE_1 t} \sin^2\theta_o + e^{-iE_2 t} \cos^2\theta_o] \bar{K}^o(0) + \\ &\quad + [e^{-iE_1 t} - e^{-iE_2 t}] \sin\theta_o \cos\theta_o K^o(0). \end{aligned} \quad (18)$$

The probability that meson  $K^o$  produced at moment  $t = 0$  will be at moment  $t \neq 0$  in the state of  $\bar{K}^o$  meson is given by a squared absolute value of the amplitude in (18)

$$\begin{aligned} P(K^o \rightarrow \bar{K}^o) &= |(\bar{K}^o(0) \cdot K^o(t))|^2 = \\ &= \frac{1}{2} \sin^2 2\theta_o [1 - \cos((E_2 - E_1)t)] \equiv \frac{1}{2} [1 - \cos((E_2 - E_1)t)], \end{aligned} \quad (19)$$

where  $\theta_o = \pi/4$ . Using expressions for masses of  $K_1^o, K_2^o$  mesons, we obtain

$$m_{K_1^o} = m_{K^o} - \Delta, m_{K_2^o} = m_{K^o} + \Delta, \quad (20)$$

where  $\Delta = 2m_{K^o \bar{K}^o}$ . Since  $m_{K^o} \gg \Delta$ ,

$$\begin{aligned} E_1 &= \sqrt{p^2 + m_{K_1^o}^2} \cong E_{K^o} (1 - \frac{m_{K^o} \Delta}{E_{K^o}^2}), \\ E_2 &= \sqrt{p^2 + m_{K_2^o}^2} \cong E_{K^o} (1 + \frac{m_{K^o} \Delta}{E_{K^o}^2}), \end{aligned} \quad (21)$$

$$E_2 - E_1 = \frac{2m_{K^o} \Delta}{E_{K^o}} = \frac{2\Delta}{\gamma}, \quad (22)$$

Then the length  $L_{12}$  of  $K^o, \bar{K}^o$  meson oscillations is as follows:

$$L_{12} = \frac{\gamma}{2\Delta} \equiv \frac{2\pi\hbar c\gamma}{2\Delta}. \quad (23)$$

The mixing angle  $\theta_o$  of  $K^o, \bar{K}^o$  is equal to  $\pi/4$  and the value for  $\Delta$  computed in the framework of weak interactions (see references in [13, 14]) is in a reasonable agreement with the same value obtained in experiments. So, we see that the scheme of mass mixings is in a rather good agreement with the experiment.

Now let us consider neutrino oscillations in the framework of the mass mixings scheme.

### 3 Vacuum mixings and oscillations of $\nu_e, \nu_\mu, \nu_\tau$ neutrinos in the scheme of mass mixings

As it is mentioned above we will work in the mass mixings scheme. We can use the  $3 \times 3$  mass matrix but since the lengths of neutrino oscillations are noticeably different, then it is proper to work by using three  $2 \times 2$  mass matrices corresponding to  $\nu_e \rightarrow \nu_\mu$ ,  $\nu_\mu \rightarrow \nu_\tau$  and  $\nu_e \rightarrow \nu_\tau$  mixings and oscillations. Since neutrino oscillations are considered in analogy with  $K^o$  meson oscillations, then we will use the method analogous to the one considered above.

#### 3.1 The case of $\nu_e, \nu_\mu$ neutrino oscillations

If the  $\nu_e, \nu_\mu$  neutrino states are produced in the weak interactions (i.e, they are eigenstates of these interactions, then the mass matrix of  $\nu_e, \nu_\mu$  - neutrinos will have a diagonal form. Following the traditions we will consider the  $\nu_e, \nu_\mu$  - neutrino mixings and oscillations by using the mass matrix and for convenience the masses are used in the linear but not in the quadratic form, then the mass matrix has the following form:

$$\begin{pmatrix} m_{\nu_e} & 0 \\ 0 & m_{\nu_\mu} \end{pmatrix}. \quad (24)$$

Since there is a interaction violating lepton numbers this mass matrix (24) becomes a nondiagonal matrix:

$$\begin{pmatrix} m_{\nu_e} & m_{\nu_e\nu_\mu} \\ m_{\nu_e\nu_\mu} & m_{\nu_\mu} \end{pmatrix}. \quad (25)$$

For obtaining the eigenstates of weak interactions which violate lepton numbers, we have to diagonalize this matrix by turning on angle  $\theta$  ( $\theta \equiv \theta_{12}$ ). By using this procedure, we get

$$U^{-1} \begin{pmatrix} m_{\nu_e} & m_{\nu_e\nu_\mu} \\ m_{\nu_e\nu_\mu} & m_{\nu_\mu} \end{pmatrix} U = \begin{pmatrix} m_{\nu_1} & 0 \\ 0 & m_{\nu_2} \end{pmatrix}, \quad U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

$$tg2\theta = \frac{2m_{\nu_e\nu_\mu}}{|m_{\nu_e} - m_{\nu_\mu}|},$$

$$\sin2\theta = \frac{2m_{\nu_e\nu_\mu}}{\sqrt{(m_{\nu_e} - m_{\nu_\mu})^2 + (2m_{\nu_e\nu_\mu})^2}}, \quad (26)$$

$$m_{1,2} \equiv m_{\nu_1\nu_2} = \frac{1}{2} \left[ (m_{\nu_e} + m_{\nu_\mu}) \pm \left( (m_{\nu_e} - m_{\nu_\mu})^2 + 4m_{\nu_e\nu_\mu}^2 \right)^{1/2} \right], \quad (27)$$

where  $\nu_1$  and  $\nu_2$  states are eigenstates of the weak interactions violating lepton numbers. Now these states are superposition states of  $\nu_e, \nu_\mu$  neutrinos.

$$\begin{aligned} \nu_1 &= \cos\theta\nu_e - \sin\theta\nu_\mu, \\ \nu_2 &= \sin\theta\nu_e + \cos\theta\nu_\mu, \end{aligned} \quad (28)$$

and the inverse transformation gives:

$$\begin{aligned} \nu_e &= \cos\theta\nu_1 + \sin\theta\nu_2, \\ \nu_\mu &= -\sin\theta\nu_1 + \cos\theta\nu_2, \end{aligned} \quad (29)$$

Then  $\nu_e, \nu_\mu$  neutrino masses are connected with masses of  $\nu_1, \nu_2$  neutrinos via the following expressions

$$\begin{aligned} m_{\nu_e} &= m_1 \cos^2 \theta + m_2 \sin^2 \theta, \\ m_{\nu_\mu} &= m_1 \sin^2 \theta + m_2 \cos^2 \theta, \end{aligned} \quad (30)$$

Now mass Lagrangian of two neutrinos ( $\nu_e, \nu_\mu$ ) has the following form:

$$\begin{aligned} \mathcal{L}_M &= -\frac{1}{2} \left[ m_{\nu_e} \bar{\nu}_e \nu_e + m_{\nu_\mu} \bar{\nu}_\mu \nu_\mu + m_{\nu_e\nu_\mu} (\bar{\nu}_e \nu_\mu + \bar{\nu}_\mu \nu_e) \right] \equiv \\ &\equiv -\frac{1}{2} (\bar{\nu}_e, \bar{\nu}_\mu) \begin{pmatrix} m_{\nu_e} & m_{\nu_e\nu_\mu} \\ m_{\nu_e\nu_\mu} & m_{\nu_\mu} \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}, \end{aligned} \quad (31)$$

while diagonalizing it transforms into the following one:

$$\mathcal{L}_M = -\frac{1}{2} [m_1 \bar{\nu}_1 \nu_1 + m_2 \bar{\nu}_2 \nu_2], \quad (32)$$

Expression for  $\sin^2 2\theta$  can be obtained from (30) and it is given by the following expression ( $\theta$  is the vacuum angle of mixing):

$$\sin^2(2\theta) = \frac{(2m_{\nu_e \nu_\mu})^2}{(m_{\nu_e} - m_{\nu_\mu})^2 + (2m_{\nu_e \nu_\mu})^2}. \quad (33)$$

This expression has the analogy with a well known formula of Breit-Wigner [16] for the transition probability:

$$W(m, m_o, \Gamma, \dots) = \frac{c(\frac{\Gamma}{2})^2}{(m - m_o)^2 + (\frac{\Gamma}{2})^2}, \quad (34)$$

where  $c$  - is a normalized factor,  $m, m_o$  are masses and  $\Gamma$  is the width transition. We can obtain expression (33) from Breit-Wigner formula if we fulfill the following substitutions

$$m \rightarrow m_{\nu_e}, \quad m_o \rightarrow m_{\nu_\mu}, \quad \frac{\Gamma}{2} \rightarrow 2m_{\nu_e \nu_\mu} \quad c \rightarrow 1. \quad (35)$$

We see that nondiagonal mass term  $2m_{\nu_e \nu_\mu}$  can be interpreted as half-width of  $\nu_e \leftrightarrow \nu_\mu$  transitions.

The evolution of  $\nu_1, \nu_2$  neutrino states with masses  $m_1, m_2$  will be given with the following expressions:

$$\nu_1(t) = e^{-iE_1 t} \nu_1(0), \quad \nu_2(t) = e^{-iE_2 t} \nu_2(0), \quad (36)$$

where  $E_k^2 = (p^2 + m_k^2)$ ,  $k = 1, 2$ .

If these neutrinos are moving without interactions, then

$$\begin{aligned} \nu_e(t) &= \cos\theta e^{-iE_1 t} \nu_1(0) + \sin\theta e^{-iE_2 t} \nu_2(0), \\ \nu_\mu(t) &= -\sin\theta e^{-iE_1 t} \nu_1(0) + \cos\theta e^{-iE_2 t} \nu_2(0). \end{aligned} \quad (37)$$

Using expression (36) for  $\nu_1$  and  $\nu_2$  and putting them in (37), we obtain

$$\begin{aligned} \nu_e(t) &= [e^{-iE_1 t} \cos^2\theta + e^{-iE_2 t} \sin^2\theta] \nu_e(0) + \\ &+ [e^{-iE_1 t} - e^{-iE_2 t}] \sin\theta \cos\theta \nu_\mu(0), \end{aligned} \quad (38)$$



$$\begin{aligned}\nu_\mu(t) = & \left[ e^{-iE_1 t} \sin^2 \theta + e^{-iE_2 t} \cos^2 \theta \right] \nu_e(0) + \\ & + \left[ e^{-iE_1 t} - e^{-iE_2 t} \right] \sin \theta \cos \theta \nu_\mu(0).\end{aligned}$$

The probability that neutrino  $\nu_e$  produced at moment  $t = 0$  will be at moment  $t \neq 0$  in the state of  $\nu_\mu$  neutrino, is given by a squared absolute value of the amplitude in (38)

$$\begin{aligned}P(\nu_e \rightarrow \nu_\mu) = & \left| (\nu_\mu(0) \cdot \nu_e(t)) \right|^2 = \\ = & \frac{1}{2} \sin^2 2\theta [1 - \cos((E_2 - E_1)t)] = \sin^2(2\theta) \sin^2((m_2^2 - m_1^2)/2p)t.\end{aligned}\quad (39)$$

The probability of  $\nu_e \rightarrow \nu_e$  is given by the following expression:

$$P(\nu_e \rightarrow \nu_e) = 1 - \sin^2(2\theta) \sin^2((m_2^2 - m_1^2)/2p)t, \quad (40)$$

where  $p^2 \gg m_1^2, m_2^2$ , and  $E_1 = \sqrt{p^2 + m_1^2} \simeq p + \frac{m_1^2}{2p}$ ,  $E_2 \simeq p + \frac{m_2^2}{2p}$ ,

Then the length  $L_{12}$  of  $\nu_e, \nu_\mu$  neutrino oscillations is

$$L_{12} = 2\pi \frac{2p}{|m_2^2 - m_1^2|}, \quad (41)$$

or

$$L_{12} = 2\pi \frac{2p}{(m_{\nu_\mu} + m_{\nu_e}) \sqrt{(m_{\nu_\mu} - m_{\nu_e})^2 + (2m_{\nu_e \nu_\mu})^2}}. \quad (41')$$

Now let us consider the cases of  $\nu_\mu \rightarrow \nu_\tau$  and  $\nu_e \rightarrow \nu_\tau$  mixings and oscillations. We will give final expressions without a detailed consideration.

### 3.2 The case of $\nu_e, \nu_\tau$ neutrino oscillations

Nondiagonal mass matrix of  $\nu_e, \nu_\tau$  neutrinos has the following form:

$$\begin{pmatrix} m_{\nu_e} & m_{\nu_e \nu_\tau} \\ m_{\nu_e \nu_\tau} & m_{\nu_\tau} \end{pmatrix}. \quad (42)$$

After diagonalizing this matrix by turning on angle  $\beta$  ( $\beta \equiv \theta_{13}$ ) we get

$$U^{-1} \begin{pmatrix} m_{\nu_e} & m_{\nu_e \nu_\tau} \\ m_{\nu_e \nu_\tau} & m_{\nu_\tau} \end{pmatrix} U = \begin{pmatrix} m_{\nu_1} & 0 \\ 0 & m_{\nu_3} \end{pmatrix}, \quad U = \begin{pmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{pmatrix}.$$

$$\text{tg} 2\beta = \frac{2m_{\nu_e \nu_\tau}}{|m_{\nu_e} - m_{\nu_\tau}|},$$

$$\sin 2\beta = \frac{2m_{\nu_e\nu_\tau}}{\sqrt{(m_{\nu_e} - m_{\nu_\tau})^2 + (2m_{\nu_e\nu_\tau})^2}}, \quad (43)$$

$$m_{1,3} \equiv m_{\nu_1\nu_3} = \frac{1}{2} \left[ (m_{\nu_e} + m_{\nu_\tau}) \pm ((m_{\nu_e} - m_{\nu_\tau})^2 + 4m_{\nu_e\nu_\tau}^2)^{1/2} \right], \quad (44)$$

where  $\nu_1$  and  $\nu_3$  states are eigenstates of the weak interactions violating lepton numbers. Now these states are superposition states of  $\nu_e, \nu_\tau$  neutrinos:

$$\begin{aligned} \nu_1 &= \cos\beta\nu_e - \sin\beta\nu_\tau, \\ \nu_3 &= \sin\beta\nu_e + \cos\beta\nu_\tau, \end{aligned} \quad (45)$$

and the inverse transformation gives:

$$\begin{aligned} \nu_e &= \cos\beta\nu_1 + \sin\beta\nu_3, \\ \nu_\tau &= -\sin\beta\nu_1 + \cos\beta\nu_3, \end{aligned} \quad (46)$$

Now the mass Lagrangian of two neutrinos  $(\nu_e, \nu_\tau)$  has the following form:

$$\begin{aligned} \mathcal{L}_M &= -\frac{1}{2} [m_{\nu_e} \bar{\nu}_e \nu_e + m_{\nu_\tau} \bar{\nu}_\tau \nu_\tau + m_{\nu_e\nu_\tau} (\bar{\nu}_e \nu_\tau + \bar{\nu}_\tau \nu_e)] \equiv \\ &\equiv -\frac{1}{2} (\bar{\nu}_e, \bar{\nu}_\tau) \begin{pmatrix} m_{\nu_e} & m_{\nu_e\nu_\tau} \\ m_{\nu_e\nu_\tau} & m_{\nu_\tau} \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\tau \end{pmatrix} \rightarrow \\ &\rightarrow -\frac{1}{2} [m_1 \bar{\nu}_1 \nu_1 + m_3 \bar{\nu}_3 \nu_3], \end{aligned} \quad (47)$$

Expression for  $\sin^2 2\beta$  has the following form:

$$\sin^2 (2\beta) = \frac{(2m_{\nu_e\nu_\mu})^2}{(m_{\nu_e} - m_{\nu_\tau})^2 + (2m_{\nu_e\nu_\mu})^2}. \quad (48)$$

The probability that neutrino  $\nu_e$  produced at moment  $t = 0$  will be at moment  $t \neq 0$  in the state of  $\nu_\tau$  neutrino, is given by the following expression:

$$\begin{aligned} P(\nu_e \rightarrow \nu_\tau) &= |(\nu_\tau(0) \cdot \nu_e(t))|^2 = \\ &= \frac{1}{2} \sin^2(2\beta) [1 - \cos((E_3 - E_1)t)] \\ &= \sin^2(2\beta) \sin^2((m_3^2 - m_1^2)/2p)t. \end{aligned} \quad (49)$$

The probability of  $\nu_e \rightarrow \nu_e$  is given by the following expression:

$$P(\nu_e \rightarrow \nu_e) = 1 - \sin^2(2\beta) \sin^2((m_3^2 - m_1^2)/2p)t, \quad (50)$$

where  $p^2 \gg m_1^2, m_3^2$ , and  $E_1 = \sqrt{p^2 + m_1^2} \simeq p + \frac{m_1^2}{2p}$ ,  $E_3 \simeq p + \frac{m_3^2}{2p}$ ,

Then the length  $L_{13}$  of  $\nu_e, \nu_\tau$  neutrino oscillations is

$$L_{13} = 2\pi \frac{2p}{|m_3^2 - m_1^2|}, \quad (51)$$

or

$$L_{13} = 2\pi \frac{2p}{(m_{\nu_\tau} + m_{\nu_e}) \sqrt{(m_{\nu_\tau} - m_{\nu_e})^2 + (2m_{\nu_e \nu_\tau})^2}}. \quad (51')$$

### 3.3 The case of $\nu_\mu, \nu_\tau$ neutrino oscillations

The nondiagonal mass matrix of  $\nu_\mu, \nu_\tau$  neutrinos has the following form:

$$\begin{pmatrix} m_{\nu_\mu} & m_{\nu_\mu \nu_\tau} \\ m_{\nu_\mu \nu_\tau} & m_{\nu_\tau} \end{pmatrix}. \quad (52)$$

After diagonalizing this matrix by turning on angle  $\gamma$  ( $\gamma \equiv \theta_{23}$ ) we get

$$U^{-1} \begin{pmatrix} m_{\nu_\mu} & m_{\nu_\mu \nu_\tau} \\ m_{\nu_\mu \nu_\tau} & m_{\nu_\tau} \end{pmatrix} U = \begin{pmatrix} m_{\nu_2} & 0 \\ 0 & m_{\nu_3} \end{pmatrix}, \quad U = \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix}.$$

$$\tan 2\gamma = \frac{2m_{\nu_\mu \nu_\tau}}{|m_{\nu_\mu} - m_{\nu_\tau}|},$$

$$\sin 2\gamma = \frac{2m_{\nu_\mu \nu_\tau}}{\sqrt{(m_{\nu_\mu} - m_{\nu_\tau})^2 + (2m_{\nu_\mu \nu_\tau})^2}}, \quad (53)$$

$$m_{2,3} \equiv m_{\nu_2 \nu_3} = \frac{1}{2} \left[ (m_{\nu_\mu} + m_{\nu_\tau}) \pm \left( (m_{\nu_\mu} - m_{\nu_\tau})^2 + 4m_{\nu_\mu \nu_\tau}^2 \right)^{1/2} \right], \quad (54)$$

where  $\nu_2$  and  $\nu_3$  states are eigenstates of the weak interactions violating lepton numbers. Now these states are superposition states of  $\nu_\mu, \nu_\tau$  neutrinos:

$$\begin{aligned} \nu_2 &= \cos\gamma \nu_\mu - \sin\gamma \nu_\tau, \\ \nu_3 &= \sin\gamma \nu_\mu + \cos\gamma \nu_\tau, \end{aligned} \quad (55)$$

and the inverse transformation gives:

$$\begin{aligned} \nu_\mu &= \cos\gamma \nu_2 + \sin\gamma \nu_3, \\ \nu_\tau &= -\sin\gamma \nu_2 + \cos\gamma \nu_3. \end{aligned} \quad (56)$$

Now the mass Lagrangian of two neutrinos  $(\nu_\mu, \nu_\tau)$  has the following form:

$$\begin{aligned}\mathcal{L}_M &= -\frac{1}{2} [m_{\nu_\mu} \bar{\nu}_\mu \nu_\mu + m_{\nu_\tau} \bar{\nu}_\tau \nu_\tau + m_{\nu_\mu \nu_\tau} (\bar{\nu}_\mu \nu_\tau + \bar{\nu}_\tau \nu_\mu)] \equiv \\ &\equiv -\frac{1}{2} (\bar{\nu}_\mu, \bar{\nu}_\tau) \begin{pmatrix} m_{\nu_\mu} & m_{\nu_\mu \nu_\tau} \\ m_{\nu_\mu \nu_\tau} & m_{\nu_\tau} \end{pmatrix} \begin{pmatrix} \nu_\mu \\ \nu_\tau \end{pmatrix} \rightarrow \\ &\rightarrow -\frac{1}{2} [m_2 \bar{\nu}_2 \nu_2 + m_3 \bar{\nu}_3 \nu_3],\end{aligned}\quad (57)$$

Expression for  $\sin^2 2\gamma$  has the following form:

$$\sin^2 (2\gamma) = \frac{(2m_{\nu_e \nu_\mu})^2}{(m_{\nu_\mu} - m_{\nu_\tau})^2 + (2m_{\nu_\mu \nu_\tau})^2}. \quad (58)$$

The probability that neutrino  $\nu_\mu$  produced at moment  $t = 0$  will be at moment  $t \neq 0$  in the state of  $\nu_\tau$  neutrino, is given by the following expression:

$$\begin{aligned}P(\nu_\mu \rightarrow \nu_\tau) &= |(\nu_\tau(0) \cdot \nu_\mu(t))|^2 = \\ &= \frac{1}{2} \sin^2(2\gamma) [1 - \cos((E_3 - E_2)t)] \\ &= \sin^2(2\gamma) \sin^2((m_3^2 - m_2^2)/2p)t.\end{aligned}\quad (59)$$

The probability of  $\nu_\mu \rightarrow \nu_\mu$  is given by the following expression:

$$P(\nu_\mu \rightarrow \nu_\mu) = 1 - \sin^2(2\gamma) \sin^2((m_3^2 - m_2^2)/2p)t, \quad (60)$$

where  $p^2 \gg m_2^2, m_3^2$ , and  $E_2 = \sqrt{p^2 + m_2^2} \simeq p + \frac{m_2^2}{2p}$ ,  $E_3 \simeq p + \frac{m_3^2}{2p}$ ,

Then the  $L_{12}$  of  $\nu_\mu, \nu_\tau$  neutrino oscillations length is:

$$L_{23} = 2\pi \frac{2p}{|m_3^2 - m_2^2|}, \quad (61)$$

or

$$L_{23} = 2\pi \frac{2p}{(m_{\nu_\tau} + m_{\nu_\mu}) \sqrt{(m_{\nu_\tau} - m_{\nu_\mu})^2 + (2m_{\nu_\mu \nu_\tau})^2}}. \quad (61')$$

Now we analyze the modern experimental data on neutrino oscillations by using the expressions obtained in the scheme of mass mixings.

## 4 Analysis of the modern experimental data on neutrino oscillations by using the results obtained in the scheme of mass mixings

### 4.1 Analysis of $\nu_e, \nu_\mu$ processes

The process of  $\nu_e, \nu_\mu$  oscillations was studied in experiment KamLAND [11] and they obtained

$$\begin{aligned} \theta \equiv \theta_{12} &\simeq 34^\circ, \quad \sin^2(2\theta_{12}) \simeq 0.860, \\ |m_2^2 - m_1^2| &= 7.50(+0.19 - 0.20) \times 10^{-5} eV^2. \end{aligned} \quad (62)$$

Using expression (33) for  $\sin^2(2\theta)$ ,

$$\sin^2(2\theta) \equiv \sin^2(2\theta_{12}) = \frac{(2m_{\nu_e\nu_\mu})^2}{(m_{\nu_e} - m_{\nu_\mu})^2 + (2m_{\nu_e\nu_\mu})^2} \simeq 0.860, \quad (63)$$

we can do the following conclusion:

$$(m_{\nu_e} - m_{\nu_\mu})^2 \simeq 0.163(2m_{\nu_e\nu_\mu})^2, \quad (2m_{\nu_e\nu_\mu})^2 \simeq 6.14(m_{\nu_e} - m_{\nu_\mu})^2$$

i.e., the difference between masses of  $\nu_e$  and  $\nu_\mu$  neutrinos is less than the nondiagonal mass term. Then deposit of  $\nu_e, \nu_\mu$  neutrino mass difference in the length of  $\nu_e, \nu_\mu$  neutrino oscillations is very small (see expr. (41')):

$$\begin{aligned} L_{12} = 2\pi \frac{2p}{|m_2^2 - m_1^2|} &\equiv 2\pi \frac{2p}{(m_{\nu_\mu} + m_{\nu_e})\sqrt{(m_{\nu_\mu} - m_{\nu_e})^2 + (2m_{\nu_e\nu_\mu})^2}} \approx \\ &\approx 2\pi \frac{2p}{(m_{\nu_\mu} + m_{\nu_e})\sqrt{(2m_{\nu_e\nu_\mu})^2}}, \end{aligned} \quad (64)$$

i.e., the length of  $\nu_e, \nu_\mu$  neutrino oscillations is mainly formed by the nondiagonal mass term  $2m_{\nu_e\nu_\mu}$ .

### 4.2 Analysis of $\nu_\mu, \nu_\tau$ processes

The process of  $\nu_e, \nu_\mu$  oscillations was studied in experiment Super-Kamiokande [15] and they obtained

$$\gamma \equiv \theta_{23} \simeq 45^\circ, \quad \sin^2(2\theta_{23}) \simeq 1.0, \quad (65)$$

and  $m_3^2 - m_2^2 = 2.1 \times 10^{-3} eV^2$ .

Using expression (62) for  $\sin^2(2\gamma)$ :

$$\sin^2(2\gamma) \equiv \sin^2(2\theta_{23}) = \frac{(2m_{\nu_\mu\nu_\tau})^2}{(m_{\nu_\mu} - m_{\nu_\tau})^2 + (2m_{\nu_\mu\nu_\tau})^2} \simeq 1.0, \quad (66)$$

we can do the following conclusion:

$$(m_{\nu_\mu} - m_{\nu_\tau})^2 \simeq 0.0, \quad m_{\nu_\mu} \simeq m_{\nu_\tau},$$

i.e., the mass of  $\nu_\mu$  neutrino is about the  $\nu_\tau$  neutrinos mass. Then the deposit of  $\nu_\mu, \nu_\tau$  neutrino mass difference in the length of  $\nu_e, \nu_\mu$  neutrino oscillations is about zero (see expression. (65), (61'))

$$L_{23} = 2\pi \frac{2p}{|m_3^2 - m_2^2|} \equiv 2\pi \frac{2p}{(m_{\nu_\mu} + m_{\nu_\tau})(2m_{\nu_\mu\nu_\tau})} = 2\pi \frac{2p}{(2m_{\nu_\mu})(2m_{\nu_\mu\nu_\tau})}, \quad (67)$$

i.e., the length of  $\nu_\mu, \nu_\tau$  neutrino oscillations is mainly formed by the nondiagonal mass term  $2m_{\nu_\mu\nu_\tau}$ . It is interesting to remark that if to suppose that  $2m_{\nu_\mu} \approx 1eV$  then from the expression  $2m_{\nu_\mu}2m_{\nu_\mu\nu_\tau} = 2.1 \times 10^{-3}eV^2$  we get  $m_{\nu_\mu\nu_\tau} \approx 10^{-3}eV$ . This value is very big in contrast to the  $K^0$  meson oscillation case where the analogous term  $m_{K^0\bar{K}^0} \approx 10^{-6}eV$ .

### 4.3 Analysis of $\nu_e, \nu_\tau$ processes

The process of  $\nu_e, \nu_\mu$  oscillations was studied in experiment KamLAND [11] and they obtained

$$\beta \equiv \theta_{13} \leq 13^\circ, \quad \sin^2(2\theta_{13}) \leq 0.192, \quad (68)$$

$|m_3^2 - m_1^2|$  is still unknown until now. Using expression (68) for  $\sin^2(2\beta)$

$$\sin^2(2\beta) \equiv \sin^2(2\theta_{13}) = \frac{(2m_{\nu_e\nu_\tau})^2}{(m_{\nu_e} - m_{\nu_\tau})^2 + (2m_{\nu_e\nu_\tau})^2} \simeq 0.192, \quad (69)$$

we can do the following conclusion:

$$(m_{\nu_e} - m_{\nu_\tau})^2 \simeq 4.21(2m_{\nu_e\nu_\mu})^2, \quad (2m_{\nu_e\nu_\mu})^2 \simeq 0.238(m_{\nu_e} - m_{\nu_\tau})^2,$$

i.e., the difference between the masses of  $\nu_e$  and  $\nu_\tau$  neutrinos is much more than the nondiagonal mass term. Then the deposit of  $\nu_e, \nu_\mu$

neutrino mass difference in the length of  $\nu_e, \nu_\tau$  neutrino oscillations are very big (see expression. (51')):

$$L_{13} = 2\pi \frac{2p}{|m_2^3 - m_1^2|} \equiv 2\pi \frac{2p}{(m_{\nu_\tau} + m_{\nu_e})\sqrt{(m_{\nu_\tau} - m_{\nu_e})^2 + (2m_{\nu_e\nu_\tau})^2}} \approx$$

$$\approx 2\pi \frac{2p}{(m_{\nu_\tau} + m_{\nu_e})\sqrt{(m_{\nu_\tau} - m_{\nu_e})^2}}, \quad (70)$$

i.e., the length of  $\nu_e, \nu_\tau$  neutrino oscillations is mainly formed by the mass difference of these neutrinos.

#### 4.4 Remarks about the problem in neutrino oscillations processes in the framework of the mass mixings scheme

If angle mixings  $\theta_{23}$  of  $\nu_\mu, \nu_\tau$  at neutrino oscillations are maximal, i.e.,  $\pi/4$  [11], then in the framework of the mass mixings scheme the masses of  $\nu_\mu, \nu_\tau$  neutrinos have to be nearly equal, i.e.,  $m_\mu \simeq m_\tau$ . Farther in the framework of this approach the equality of oscillation lengths of  $L_{23}$  and  $L_{13}$  is impossible to obtain without additional supposition (see expressions (51'), (61')), in contrast to the conventional approach, since the nondiagonal mass components  $m_{\nu_\mu\nu_\tau}, m_{\nu_e\nu_\tau}$  of the mass matrix cannot be equal by definition. Then the natural solution of this problem is to suppose that  $L_{13}$  is larger than  $L_{23}$ , then the length of neutrino oscillations  $L_{13}$  has to be larger than  $L_{23}$ , i.e., the value of  $\theta_{13}$  is necessary to search on distances more than  $L_{23}$ .

To solve this problem in the framework of the standard approach [10], it is necessary to suppose that

$$(m_2^2 - m_1^2) \neq (m_3^2 - m_1^2) - (m_3^2 - m_2^2). \quad (71)$$

Obviously, it is possible if to suppose that 4 neutrino oscillations are realized instead of 3 neutrino oscillations, i.e., if there is the fourth component.

## 5 Conclusion

On the example of  $K^o$  mixings and oscillations we have considered mixings and oscillations of  $\nu_e, \nu_\mu, \nu_\tau$  neutrinos. The analysis of the

obtained results has been done by using modern experimental data on neutrino oscillations. In these experimental data there is one problem. If we use the conventional theoretical approach [10], then  $L_{13} \approx L_{23}$  ( $L_{12} \gg L_{23}$ ) and the mixing angle of  $\theta_{13}$  is very small. However, angle mixings  $\theta_{23}, \theta_{12}$  are big and close to maximal angle  $\pi/4$ . The problem is: why is mixing angle  $\theta_{13}$  so small?

Since the angle mixings  $\theta_{23}$  of  $\nu_\mu, \nu_\tau$  at neutrino oscillations is maximal [12], i.e.,  $\pi/4$  then in the framework of the mass mixings scheme the masses of  $\nu_\mu, \nu_\tau$  neutrinos have to be nearly equal, i.e.,  $m_\mu \simeq m_\tau$ . In the framework of this approach the equality of oscillation lengths of  $L_{23}$  and  $L_{13}$  is impossible to obtain without additional supposition (see expressions (51'), (61')) since the nondiagonal mass components  $m_{\nu_\mu \nu_\tau}, m_{\nu_e \nu_\tau}$  of the mass matrix cannot be equal by definition. Then the natural solution of this problem is to suppose that  $L_{13}$  is larger than  $L_{23}$  ( $L_{13} > L_{23}$ ). Then it is necessary to examine  $\nu_e, \nu_\tau$  neutrino oscillations at much longer distances than  $L_{23}$  to search for the value of  $\theta_{13}$ .

To solve this problem in the framework of the standard approach [10], it is necessary to suppose that  $(m_2^2 - m_1^2) \neq (m_3^2 - m_1^2) - (m_3^2 - m_2^2)$ . Obviously, it is possible if to suppose that 4 neutrino oscillations are realized instead of 3 neutrino oscillations, i.e., if there is the fourth component.

## Список литературы

- [1] Gell-Mann M., Pais A., Phys. Rev., 1955, v.97, p.1387;  
Pais A., Piccioni O., Phys. Rev., 1955, v.100, p.1487;  
Okun L. B., Weak interactions of elementary particles,  
Fismatizdat, M, 1963.
- [2] Treiman S. B., Sachs R. S., Phys. Rev., 1956, v.103, p.1545.
- [3] Beshtoev Kh. M., IL Nuovo Cim.A, 1995, v.168, p.275.
- [4] Beshtoev Kh. M., JINR Rapid Communications N3(71)-95;  
The Intern. Sym. on Weak and Electrom. Inter. in Nuclei, 1995,



June, Osaka, Japan, p.15.

- [5] Beshtoev Kh. M., Proc. of 24-th Intern. Cosmic ray Conf., Rome, 1995. vol.4, p.1237;
- [6] Beshtoev Kh. M., Proceedings of 4-th Int. School "Particles and Cosmology Baksan, 1995 (World Scient., Singapoure, P.290)
- [7] B. M. Pontecorvo , Soviet Journ. JETP, 33 (1957) 549;  
JETP, 34 (1958) 247.
- [8] Z. Maki et al., Prog.Theor. Phys., 28 (1962) 870.
- [9] B. M. Pontecorvo, Soviet Journ. JETP, 53 (1967) 1717;  
Gribov V., Pontecorvo B.M., Phys. Lett.B 1969, v.28, p.493.
- [10] Phys. Lett. B, Rev. of Particle Phys., 2008, v.667, p.530.
- [11] Gando A., et al., hep-ex/1009.4771v.3.
- [12] Takeuchi Y., hep-ex/1112.3425v.1.
- [13] Commins E.D., Bucksbaum P.H., Weak Interactions of Leptons and Quarks, Cambridge ..., 1983.
- [14] Okun L. B., Lepton and Quarks, M., Nauka, 1990.
- [15] Cabibbo N., Phys. Rev. Lett., 1963, v.10, p.531;  
Kobayashi M., Maskawa K., Prog. Theor. Phys., 1973, v.49, p.652.
- [16] Weisskopf V. and Wigner E.P., Z. Phus., 1930, v.63, p.54.